05/10/15

Φυσικοί Αριθμοί

Αξιώματα του Peano

1) $\exists\, \mathbb{N} \neq \emptyset$
2) $1 \in \mathbb{N}$
3) $\exists\, S : \mathbb{N} \to \mathbb{N}$
4) $\nexists\, n \in \mathbb{N} : 1 = S(n)$
5) $P(n)$: i) $P(1)$ αληθής
   ii) αν $P(k)$: αληθής $\Rightarrow P(S(k))$ αληθής $\Big\} \Rightarrow$ $P(n)$ αληθής $\forall n \in \mathbb{N}$.

Ανισότητα Bernoulli
$(1+\alpha)^n \geq 1 + n\cdot\alpha$
$n \in \mathbb{N}$
$\alpha \geq 1$, ~~…~~ $\alpha \in \mathbb{R}$

Πρόσθεση : $S(n) = n+1$
$S(n+1) = n+2 = S(S(n))$

Πολ/μός : $n \cdot 1 = n$
$n \cdot S(m) = n \cdot m + n$

Διάταξη : $m > n$: $\exists k \in \mathbb{N}$: $m = n + k$

$A, \sigma \subseteq A \times A$
σ : συνάρτηση
αν $(\alpha, b), (\alpha', b') \in \sigma$
κ' $\alpha = \alpha' \Rightarrow b = b'$

$x + 3 = 2$ Ακέραιοι

$\mathbb{Z} = \{0\} \cup \mathbb{N} \cup \{-x : x \in \mathbb{N}\}$

$\mathbb{N} \times \mathbb{N} = \{(n, m) : n, m \in \mathbb{N}\}$

$(n, m) \simeq (n', m')$: $n + m' = n' + m$ (σχέση ισοδυναμίας)

Ρητοί : $\mathbb{Q} = \{(\alpha, b) : \alpha \in \mathbb{Z}, b \in \mathbb{N}\}$

σ : σχ. διάταξης
αν $(\alpha, b), (b, c) \in \sigma \Rightarrow$
$\Rightarrow (\alpha, c) \in \sigma$
κ' $(\alpha, b), (b, \alpha) \in \sigma \Rightarrow \alpha = b$

$(\alpha, b) \approx (\alpha', b')$: $\alpha b' = b \cdot \alpha'$

$[(\alpha, b)] = \frac{\alpha}{b}$

$(\alpha, b) + (\alpha', b') = (\alpha b' + \alpha' b, b b') = (\alpha\alpha', b b')$

$\mathbb{Q}$, $\mathbb{R}$.

$r_1, \ldots, r_n, \ldots \to 0$

$\forall \varepsilon \in \mathbb{Q}^+ : \exists n_0 : |r_n| < \varepsilon, \forall n \geq n_0$

$A = \{(r_n) : (r_n) \text{ βασική}\}$

$(r_n) \approx (s_n) \quad : (r_n - s_n) \in \mathcal{J}$, με $\mathcal{J}$: το σύνολο των μηδενικών ακολουθιών.

$r_n - s_n \to 0$

$(+)\ s_n - t_n \to 0$ $\qquad [(r_n)] = \alpha \ (\in \mathbb{R})$

$r_n - t_n \to 0$

$$\underset{\omega}{\mathbb{N}} < \underset{\alpha}{2^{\mathbb{N}}} \approx \mathbb{R} < \underset{\alpha+1}{2^{\mathbb{R}}} < \underset{\alpha+2}{2^{2^{\mathbb{R}}}} < \underset{\alpha+3}{2^{2^{2^{\mathbb{R}}}}}$$

$2^{\mathbb{N}} = \{0,1\}$

$x + (\pm\infty) = \pm\infty \quad (\forall x \in \mathbb{R}) \quad = (\pm\infty) + x$

$$x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty, & x > 0 \\ \mp\infty, & x < 0 \\ 0, & x = 0 \end{cases}$$

$\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$

$(x_n) \in \mathbb{R}$

$s_k = \sup\{x_n : n \geq k\} \in \mathbb{R}^*$

$\limsup x_n = \inf\{s_k : k \in \mathbb{N}\}$

$$\boxed{\limsup x_n = \inf_{k} \sup_{n \geq k} x_n}$$

$$x_n = \begin{cases} \frac{1}{n}, & n = 3\lambda \\ e^n, & n = 3\lambda+1 \\ 2-n, & n = 3\lambda+2 \end{cases}$$

$x_1 = e \quad x_2 = 0 \quad x_3 = 1 \quad x_4 = e^4$

$x_5 = -3, \ x_6 = 1/6, \ x_7 = e^7, \ldots$

$s_1 = +\infty$

$s_2 = +\infty, \quad \forall k \in \mathbb{N} \quad s_k = +\infty \quad$ άρα $\limsup x_n = +\infty$

$i_n = \inf\{x_n : n \geq k\}$

$$\boxed{\liminf x_n = \sup_{k} \inf_{n \geq k} x_n}$$

$i_k = -\infty$, $\forall k \in \mathbb{N}$ άρα $\liminf x_n = -\infty$

$\limsup(-x_n) = \inf_k \sup_{n \geq k}(-x_n) =$

$= \inf_k \left(-\inf_{n \geq k} x_n\right) = -\sup_k \inf_{n \geq k} x_n =$

$= -\liminf x_n$

δηλ. $\boxed{\limsup(-x_n) = -\liminf(x_n)}$

ΚΑΙ $\boxed{\liminf(-x_n) = -\limsup x_n}$

$A \subseteq \mathbb{R}$

$-A = \{x : -x \in A\}$

$\sup(-A) = b$

i) $x \leq b$, $\forall x \in (-A)$

ii) $x \leq c$, $\forall x \in (-A) \Rightarrow b \leq c$

1) $-x \geq -b$ $\forall (-x) \in A$

$\quad x'$, $\forall x' \in A$

2) $x' \geq -c$, $\forall x' \in A \Rightarrow$

$\Rightarrow -b \leq -c$

άρα $-b = \inf A$

άρα $b = -\inf A$

Θδο $\liminf x_n \leq \limsup x_n$

Έστω k, k' $k_0 = \max\{k, k'\}$

τότε $\inf\{x_n : n \geq k\} \leq \inf\{x_n : n \geq k_0\} \leq$

$\leq \sup\{x_n : n \geq k_0\} \leq \sup\{x_n : n \geq k'\}$

δηλαδή $i_k \leq s_{k'} \Rightarrow \sup i_k \leq s_{k'}$, $\forall k' \Rightarrow$

$\Rightarrow \boxed{\sup i_k \leq \inf s_{k'}}$

$\quad \|$

$\boxed{\liminf x_n \leq \limsup x_n}$